REAL ANALYSIS PRELIMINARY EXAM
AUGUST, 2025

All responses require justification.

(1) Let
$$\omega = (2x + e^{y^2 z})dy \wedge dz + \sin(x^2 + z^2)dz \wedge dx + z^3 dx \wedge dy$$
and let Ω be the region inside of the sphere $x^2 + y^2 + z^2 = 4$ and outside the cone $x^2 + y^2 - z^2 = 0$. Find $\int_{\partial\Omega} \omega$, where $\partial\Omega$ is the boundary of the region Ω. Assume that the orientation of the boundary $\partial\Omega$ comes from the usual orientation inside Ω.

(2) Let
$$f(x,y) = \begin{cases} \frac{x^3+6y^3}{x^2+2y^2}, & \text{if } (x,y) \neq (0,0), \\ 0, & \text{if} (x,y) = (0,0). \end{cases}$$

(a) Prove that $f(x,y)$ is continuous at $(0,0)$.

(b) Is $f(x,y)$ differentiable at $(0,0)$? Please explain.

(3) Suppose that K is a subset of $\mathbf{R}^n$ such that every continuous function from K to $\mathbf{R}$ is bounded. Prove that K is compact.

(4) Let $f(x)$ be a smooth periodic function with the period 2π. Assume that for all x,
$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right).$$
Express $\int_0^{2\pi} (f(x))^2 dx$ in terms of numbers (a_n) and (b_n). Justify your reasoning.

(5) Show that if (a_n) is a decreasing sequence of real numbers such that $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} n a_n = 0$.

(6) Suppose that $f \in C^2(\mathbf{R})$, $g \in C^0(\mathbf{R})$ satisfy $f''(x) + g(x)f'(x) - f(x) = 0$ for all $x \in \mathbf{R}$, and that $f(0) = f(1) = 0$. Prove that f vanishes identically on $[0,1]$. (HINT: What can you say about the extrema of a function on a closed interval?)

(7) Prove that the function
$$f(x) = \frac{\sin(x^3)}{x}$$
is uniformly continuous on $(0, \infty)$.

(8) Compute
$$\lim_{n\to\infty} \int_0^{\infty} \frac{x}{\sqrt{1+x^n}} dx.$$
Justify your computations.